Precession of spacetime geodesics in Schwarzschild's spacetime

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Summary The precession of bounded spacetime orbits in Schwarzschild's spacetime is well-known topic.[1] In this article, we just recall the method and its main results.

1 Geodesic equation in Schwarzschild's metric

Original Schwarzschild's metric is recalled in equations (3) below.

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 \ + \sin^2\theta d\varphi^2) \tag{1}$$

In spacetime, as we will consider only timelike geodesics, the geodesic equation is usually written [2]:

$$-E^2 + \frac{dr^2}{d\tau^2} + (1 - \frac{2GM}{r})(\frac{L^2}{r^2} + 1) = 0 \tag{2}$$

In equation (2), per the spherical symmetry, we set $\theta = \pi/2$ [3] and we introduced, for a unit mass, the constants of motion E in equation (2) and $L = r^2 d\varphi/d\tau$ in equation (2).

2 The analytic solution in spacetime

By multiplying equation (4) by $(d\tau/d\varphi)^2 = r^4/L^2$ we get:

$$\frac{dr^2}{d\varphi^2} = E^2\frac{r^4}{L^2} - (1 - \frac{2GM}{r})(\frac{r^4}{L^2} + r^2) \tag{3}$$

With $r = 1/u$, we get:

$$(\frac{du}{d\varphi})^2 = \frac{E^2}{L^2} - (1 - 2GMu)(\frac{1}{L^2} + u^2) = 2GMu^3 - u^2 + \frac{2GM}{L^2} - \frac{1 - E^2}{L^2} = 2GM(u - u_1)(u - u_2)(u - u_3) \tag{4}$$

Where u_1, u_2, u_3 are the roots of the cubic polynomial $au^3 + bu^2 + cu + d$ with real coefficients $a = 2GM, b = -1, c = 2GM/L^2, d = -(1 - E^2)/L^2$, as defined above.[4]

By setting $v = u - b/3a$ and dividing by a, we get a polynomial in v, $v^3 + p.v + q$:

$$v^3 + (\frac{1}{L^2} - \frac{1}{12GM^2})v + \frac{3E^2 - 2}{6GML^2} - \frac{1}{108GM^3} \tag{5}$$

Where $p = 1/L^2 - 1/(12GM^2)$ and $q = (3E^2 - 2)/(6GML^2) - 1/(108GM^3)$. This equation is generally used for solving the equation and then by using $u = v + b/3$ we get the roots of the original polynomial.

In this equation, the sign of

$$\Delta = -(4p^3 + 27q^2) = \frac{-16GM^4 - (-8 + 36E^2 - 27E^4)GM^2L^2 + (E^2 - 1)L^4}{4GM^4L^6}) \tag{6}$$

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[1] For analytic solution solving the spacetime equation, see for instance [?], chapter 19, where one will find a quite extensive analysis of this problem.

[2] See for instance [1] equation (5.64) page 208

[3] These geodesics take place in a plane. Per this constraint, the number of degrees of freedom is reduced to 3.

[4] These roots can be calculated by using the line of command of mathematica: Solve $[2 * GM * u^3 - u^2 + 2 * GM * u/L^2 - (1 - E^2)/L^2 == 0, u]$

allows to know the kind of the roots. If $\Delta > 0$, there are three real roots $u_1 < u_2 < u_3$ and if $E^2 < 1$, a stable bounded orbit exists where perihelion is given by the root u_2 and aphelion by root u_1. The root u_3 is the aphelion of an unstable orbit ending to the central singularity. [5]. If $\Delta < 0$ there is one real root called u_3 and two complex conjugate roots labeled u_1, u_2. If $\Delta = 0$, there are multiple roots, all real. These roots define the extrema of the function $u(\varphi)$.[6]. They will define the types of the geodesics.[7].

The solution of equation (63) is an $Elliptic_F$ integral (Incomplete elliptic integral of first kind)[8].

$$\frac{\varphi}{2}\sqrt{2GM(u_3-u_1)} = \int_0^{\psi} \frac{d\theta}{\sqrt{1-(\frac{u_2-u_1}{u_3-u_1})\sin^2\theta}} = Elliptic_F[\arcsin(\sqrt{\frac{u-u_1}{u_2-u_1}}), \frac{u_2-u_1}{u_3-u_1}] \tag{7}$$

For the precession, as in the space solution, we will use the $Elliptic_K$ integral (complete elliptic integral of first kind) noted K(k). We get:[9]

$$\varphi = \frac{4 * K(\frac{u_2-u_1}{u_3-u_1})}{\sqrt{2 * GM(u_3-u_1)}} \tag{8}$$

As the parameters in this equation are $(u_2 - u_1), (u_3 - u_1)$ the precession will be the same for the class of different equations with roots $u_1 + a, u_2 + a, u_3 + a$ for a function *a (M, E, L)* depending on *M, E*, and *L*. Moreover, as there are two different parameters only, built with the roots, this means that the precession can be, as well, defined by an equation of degree 2, the roots of which are $(u_2 - u_1), (u_3 - u_1)$ which are function of p and q and therefore of *M, L, E*.

Example of precession in strong Field:

In spacetime:

With $L = 5, E^2 = 0.96 < 1$, the three roots of equation (63) are real $u_1 = 0.0319308, u_2 = 0.0616454, u_3 = 0.406424$, therefore $u_2 - u_1 = 0.0297146, u_3 - u_1 = 0.3744932$ in units where $GM = 1$ and $c = 1$. The result given by equation (67) is:

$$\varphi = \frac{\pi}{2}(1.17949) \tag{9}$$

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References

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[5] For instance three real roots exist for $L = 5$, if $0.96 < E^2 < 1.2$. With $E^2 = 0.96$, the roots are $u_1 = 0.0319308, u_2 = 0.0616454, u_3 = 0.406424$, therefore $u_2 - u_1 = 0.0297146, u_3 - u_1 = 0.3744932$ in units where $GM = 1$ and $c = 1$.

[6] And therefore those of $r(\varphi)$, as $u = 1/r$, whose extrema are of opposite type.

[7] [?], chapter 19, for a detailed description of these types.

[8] To be consistent with previous notation , we use the parameter $k^2 = \frac{u_2-u_1}{u_3-u_1}$ and not the modulus $k = \sqrt{\frac{u_2-u_1}{u_3-u_1}}$ in elliptic integrals and associated functions. In equation (66), $\theta = \arcsin\sqrt{\frac{u-u_1}{u_2-u_1}}$.

[9] Let us stress that equation (66) defines $\varphi/2$ and equation (67) defines φ, this explaining the factor 2 between them, as in the space solution.

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