Painlevé form: a quasi-flat description of Schwarzschild spacetime!

Jacques Fric, Université Paris-Diderot, Laboratoire SPHERE, Paris, France

Summary

We analyze what the relativistic form of Painlevé, first in the history to be not singular on the horizon, reveals on Schwarzschild's problem and beyond, how they contribute to the epistemological understanding of general relativity and how the key place he gives implicitly to the observer, in his preliminary statement, allows to settle the problem in a fundamentally physical context. We will demonstrate how and why the breaking concept of spacetime orientation, key point of his innovative proposal was not understood, dooming his contribution to a total failure. The examination of some original attributes exhibited by this form, will lead us to an epistemological discussion on concepts, such as the expansion or collapse of an empty spacetime and how some of its Newtonian attributes rely on a hidden anti-auto-duality revealed by this form of metric.

1 - Introductory article (10/24/1921)

In his first article (10/24/1921), [19], Painlevé convinced that general relativity is only a fashionable way to present some original aspects of Newtonian gravitation engages in a critical analysis of its philosophical and scientific claims. He continues, more constructively by pointing out that Schwarzschild's form is not the only solution to the problem as there is an infinity, depending only on two arbitrary functions of the coordinate *r*.

He will give some generic form for them, in his second article [20],(11/14/1921)[1], and proposes one of them[2], in this first paper which can be written[3]

$$ds^2 = (1-\frac{a}{r})dt^2 + 2\frac{\sqrt{a}}{r}dr.dt - (dr^2 + r^2(sin^2\theta d\varphi^2 + d\theta^2)), \qquad (1\text{-}2)$$

which, according to his opinion, seems to invalidate some claims of the Einstein's theory[4]. In his conclusion, Painlevé misinterprets the meaning of the spectral shift, this leading him to make some controversial statements about the infinitesimal line element (ds^2).

This led to discredit his proposal. We will show that, beyond his erroneous interpretation, his concluding remarks are indicative of the method used by Painlevé.

2 - Second article (11/14/1921)

Generic form, in spherical coordinates, independent of time

In the following article (11/14/1921), after proposing an original and innovative covariant geometric form, allowing to define a proper time generated by the gravitation in Newton's mechanics[5], he expands the statements made in his previous paper (10.24.1921):

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[1]This second article, written shortly after the first one, will clarify his first statement. Painlevé gives a general form defining a class of a double infinity of solutions. This statement shows that on the 10/24/1921 he had already established this generic form and likely an important part of the article that follows.

[2]Painlevé had understood that there was an infinite number of forms of the metric corresponding to the Schwarzschild's solution and that the choice of the form of the metric was arbitrary. [5] Eisenstaedt J. (1982). p 174.

[3]Gullstrand A. [10] proposed the same form independently, shortly after. Here, $a = 2GM/c^2 = 2GM$, as we generally set $c = 1$. The numbering of equations is referring to those of the articles of Painlevé with a prefix corresponding to the rank of the article. Painlevé uses the same spherical coordinates (r, θ, φ) than Schwarzschild.

[4]This form describes the expanding region "white hole" that neither Einstein nor Painlevé and his followers have identified.

[5]This form is the subject of another publication [9].

"According to the Einsteinian mechanics, the equations of motion must be part of a large, but special class of second-order equations, that define the geodesics of a four variables line element, ds², such as:

$$ds^2 = A(r)dt^2 - 2B(r)dtdr - C(r)[r^2(d\theta^2 + sin^2\theta d\varphi)] - D(r)dr^2 \quad (2\text{-}3)$$ [6]

For r = ∞, according to the principle of inertia and to the axiom of Fresnel, we must have A = V², 7 B = 0, C = D = 1, V denoting the velocity of light, far away from any matter field. By a suitable choice of units, we assume V = 1."

Painlevé points out that the covariance constraint, so often invoked, constrain only the form of equations and is in fact a "truism" because any reasonable theory can comply with it, therefore he will constrain his form *(2-3)* by using the Einstein equation.

Painlevé defines a generic metric with a double infinity of functions which is a solution of Einstein's equation

Then, Painlevé declares:

"Whatever the functions A(r), B(r), C(r), D(r), that the experience would lead us to adopt, it is always possible to form some invariant conditions that the coefficients of ds² must satisfy, when one replaces r, θ , φ and t, by functions of arbitrary four variables.

But, a priori, Einstein claims that these invariant conditions should be ruled by the partial second order derivatives of a special form, which depends on both the theories of Newtonian gravitation in curvilinear coordinates, and of the theory of curvature of ordinary surfaces.[8].
This is these stringent restrictions and not the truism of covariance which, among the possible ds ² of the form (2-3), selects the following:

$$ds^2 = (1 - \frac{2\mu}{f(r)})[dt - \chi(r)dr]^2 - f^2(r)(d\theta^2 + sin^2\theta d\varphi^2) - \frac{f'^2(r)dr^2}{1-\frac{2\mu}{f(r)}} \quad (2\text{-}4)$$

Where μ is a constant and f(r) and χ(r) are two arbitrary functions of r only, such that χ (r) tends to zero and f '(r) (always positive) tends to 1 when r tends to the infinity. "

Equation (2-4) *satisfies identically the Einstein's equation in vacuum* for any pair of functions *f (r)* and *χ (r).* This defines a double infinite class of metric[9], in spherical coordinates *(t, r, θ, φ).*

3 -Objectivity of the necessary oriented phenomenology of the solution

This is the key point of his solution that was totally misunderstood by the scientific community, including Einstein and Painlevé himself who will constrain unduly his equation (2-4) by the "reversibility" postulate *V* of the second article, stating that the *ds²* should be invariant under a *t* to *-t* transformation, leading him to declare that only quadratic terms were allowed in the metric!
The oriented character of the horizon, which can be crossed inward only, yields a physically oriented phenomenology.

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[6]There is a typo in the original text which reads: ds ² = A (r) dt ²-2B (r) dt.dr-C (r) [r ² dθ ² + sin ² θdφ ²]-D (r) dr ².
[7]There is another typo in the original text which reads: A = V, but as he set V =1, this has no consequences.
[8]Although he does not say it explicitly, Painlevé refers to Einstein's equation.
[9] For χ(r) = 0 and f(r) = r,→ f '(r) =1, one gets Schwarzschild's form. For $\chi(r) = (2M/r)^{1/2}(1-2M/r)^{-1}$ and f (r) = r, → f '(r) = 1, one gets Painlevé's form, proposed in his 1st article (24/10/1921*)* and so on for other forms. Unfortunately, this form coming from the constraint $R_{\mu\nu} = 0$, when computed with the generic metric (2-3), as stated by Painlevé, was forgotten!

Therefore, (in this type of coordinates), the metric, which represents the phenomenology, must be also oriented for formally removing the “singularity” on the horizon.

Painlevé's equation (1-2) described this asymmetry of the spacetime representation by using a ($dt.dr$) term in the infinitesimal line element ds^2, which allows to discriminate a motion defined by $dr/d\tau > 0$ from a motion defined by $dr/d\tau < 0$,[10] which is not possible, if all terms of the line element, are quadratic. This was clearly expressed by Lemaître, in his article [15], chapter 11[11].

It is instructive that this is the crucial point that has led to the rejection of the Painlevé's proposal[12].

This was confirmed at the debate on “infinite potentials on the horizon” held at the “Collège de France” on April 7th 1922, where Einstein, still concerned by this problem, ignoring Painlevé's proposal[13], discards the problem by a “demonstration” showing that an infinite pressure would occur in the center of a collapsing ball of matter, before reaching the horizon radius [18]!

Although this form seemed odd to Einstein and his contemporaries, because of the presence of a non-quadratic term, conferring locally an orientation to spacetime, this form directly derived from spherical symmetry and constrained by the Einstein equation is perfectly valid.

As a mathematician, Painlevé, could not renounce, without an overwhelming epistemological reason, to his proposal. This why he invoked (wrongly) a physical “reversibility principle” for superseding the mathematics!

4- Flat attribute revealed in the tetrad formalism

Hamilton & Lisle noticed in their paper [11] that a kind of flat global coordinates emerge from Painlevé's Cartesian coordinates

The Ricci rotation coefficients $\Gamma^k{}_{mn}$, (antisymmetric on the two first indices), which encode the curvature of spacetime, key parameter of the local geodesic equation[14], in the tetrad formalism remain identical when we replace some tetrads $e_\mu{}^m$ and inverse tetrads $e_m{}^\mu$ by corresponding Kronecker's symbols $\delta_\mu{}^m$ and $\delta_m{}^\mu$.[15]

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[10] In the form of Painlevé, as for the fiducial observer, the coordinate t is equal to τ, proper time of the fiducial observer, $dr/dt = dr/d\tau$. So, the local anisotropy is implied by the presence of a cross term ($dr.dt$).

[11] “This is because one wanted a static field that there is a singularity on the horizon which is actually fictitious”.

[12] This is related to a misunderstanding of the representations of curved spaces. The fundamental mathematical work of H. Weyl and E. Cartan, still in progress at that time, was not advanced enough to have a clear understanding of such concepts, See [23] Scholz E. (2010).

[13] Extracts from C. Normann report [18] (April 5th 1922) « It was Mr. Hadamard, celestial mechanics professor at the Collège de France, who opened fire with a question relating to the formula by which Einstein expresses the new law of universal gravitation. In this formula, under the simple form that Schwarzschild gave to it and that answers all the practical needs of astronomy, there exists a certain term that Mr. Hadamard is very much concerned with; if the denominator of that term becomes null, meaning if this term becomes infinite, the formula no longer makes sense, or at least one could demand what is its physical meaning »….

« Einstein does not hide the fact that this very profound question is somewhat embarrassing to him. “If,” he says, “this term could effectively become null somewhere in the universe, then it would be an unimaginable disaster for the theory; and it is very difficult to say *a priori* what would occur physically, be-cause the formula ceases to apply.” Is this catastrophe—which Einstein pleasantly calls the “Hadamard catastrophe” possible, and in this case what would be its physical effects? ».

[14] The geodesic equation in the tetrad frame for a 4-vector $\boldsymbol{p}$ of coordinates p^k is: $dp^k/d\tau + \Gamma^k{}_{mn}\, u^n p^m = 0$, where u^n are the components of the velocity of a radially free-fall observer.

[15] $\Gamma_{kmn} = ½ (d_{kmn} - d_{mkn} + d_{nmk} - d_{nkm} + d_{mnk} - d_{knm}) = ½ (d'_{kmn} - d'_{mkn} + d'_{nmk} - d'_{nkm} + d'_{mnk} - d'_{knm})$ where $d_{kmn} = \eta_{kl}\, e_\lambda{}^l\, e_n{}^\nu\, \partial_\nu e_m{}^\lambda$ and $d'_{kmn} = \eta_{kl}\, \delta_\lambda{}^l \delta^\nu{}_n\, \partial_\nu(\delta_m{}^\lambda + \delta_m{}^0 \beta^\lambda) = \eta_{kl}\, \delta_\lambda{}^l \delta^\nu{}_n\, \partial_\nu(\delta_m{}^0 \beta^\lambda) = \partial_n(\delta_m{}^0 \beta_k)$, d_{kmn} and d'_{kmn} are not identical but give the same Γ_{kmn}.

As the geodesic equation, in the tetrad frame coordinates, describes the same (physical) motion of free-falling objects than that in Painlevé coordinates, this confers to this equation a "flat flavor".[16] This is corroborated by the redshift phenomenology of light in the Painlevé's form (between a light ray emitted in a Painlevé's observer frame and received in another Painlevé's observer frame) which obeys to the relativistic Doppler effect [8] . As soon as in 1922, M. Sauger [22] noticed that the metric of the “Schwarzschild” problem, could be built by using only the special relativity

The comparison between $\Gamma^{\lambda}_{\mu\nu}$, Christoffel's symbols of Painlevé's Cartesian metric and Γ^{k}_{mn}, Ricci's rotation coefficients of Lorentz's frame, shows that the latter's, which reduce to $\Gamma^{0}_{ij} = \Gamma^{i}_{0j}$, are a subset of $\Gamma^{\lambda}_{\mu\nu}$. This means that the transformation, between the curved spacetime described by Painlevé's metric, and the Lorentz's spacetime described in the tetrad formalism, just consists to nullify a part of Christoffel's symbols.

The surviving symbols $\Gamma^{0}_{ij} = \Gamma^{i}_{0j}$, represent Ricci's rotation coefficients and are, in fact, the projection, on the Euclidean spatial hypersurface (orthogonal to the Painlevé's observer geodesic), of the **extrinsic curvature** called second fundamental form, a key parameter in general relativity!

5 - Vanishing of Landau-Lifchitz's pseudo-tensor.

The inverse metric

Even though the metric is widely used in general relativity when using the covariant description of the theory, in non-covariant approaches, such as the Landau-Lifchitz proposal [14] who claims that gravitation is a field in a Minkowski spacetime, the inverse metric is more appropriate. This is bound to this formalism involving a field in a flat spacetime which one assume to be bound to the local Minkowskian tangent space.

The inverse metric, in spherical coordinates, where $\eta^{\mu\nu}$ is Minkowski's tensor of inverse metric, may be written

$$\partial_s^2 = \eta^{\mu\nu}\,\partial_\mu\,\partial_\nu + \boldsymbol{\partial_t^2} + V^\mu V^\nu\,\partial_\mu\,\partial_\nu = \delta^{ij}\,\partial_i\,\partial_j - [\partial_t - (2M/r)^{1/2}\,\partial_r\,]^2,$$

where the components of the 4-velocities V^μ and V_μ, in the tangent and cotangent space are:

$$V^\mu = \{1,\, -(2M/r)^{1/2},\, 0,\, 0\,\} \quad \rightarrow \quad V_\mu = \{-1,\, 0,\, 0,\, 0\,\}.$$

As we can see, this inverse metric is the sum of the Euclidean metric and of the tensorial product of a 4-vector V^μ, (defined above) by itself. This exhibits the key importance of this vector.

This form induces a foliation of spacetime, in these coordinates. The space sections, which are Euclidean, are orthogonal to the geodesic worldlines followed by the fiducial observer, (of 4-velocity $= V^\mu$), called the Painlevé's observer.

Here we used spherical coordinates which yield a quite simple form. But, as they suffer from a number of defects (singularity for $r = 0$ and non-unitary determinant), we will use the Cartesian coordinates which are exempt of these defects.

Introduction and definition of this pseudo-tensor

Landau and Lifchitz were looking for a conservation law in their minkowskian theoretical approach.

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[16] The change in local Minkowski's basis of vectors, when moving on the geodesic, is ruled by special relativity (local boost). This surprising property is only exhibited in Cartesian Painlevé's form.

But the conservation equation of general relativity $T^{\mu\nu}{}_{;\nu} = 0$ (covariant divergence that takes into account the curvature) does not satisfy[17] the conservation ordinary equation $T^{\mu\nu}{}_{,\nu}= 0$, associated to divergence in flat spacetime.
For complying with the rules of divergence in flat spacetime they have to introduce a correcting term in the equation, for taking into account the effect of gravity. Therefore, they introduce $t^{\mu\nu}$ that will represent the difference between the two laws, such that the 4-divergence, in flat spacetime, of its sum with its energy-momentum tensor will be zero. This leads them to propose $(T^{\mu\nu}+t^{\mu\nu})$[18], in place of $T^{\mu,\nu}$ which should comply with $\partial_\nu(-g[T^{\mu\nu}+t^{\mu\nu}]) = 0.$

In this equation, g denotes the metric's determinant and $t^{\mu\nu}$ is a pseudo-tensor which represents the gravitation. This equation is not covariant because of the presence of $t^{\mu\nu}$ and of ordinary partial derivatives. Note the position of the indices which favors an approach in the tangent space.
Landau and Lifchitz (1951), [14], proposed a symmetrical solution[19] as defined in § 96 equation (96,7) of the Landau & Lifchitz (1994), [13].

$$t^{\mu\nu} = -\frac{G^{\mu\nu}}{8\pi G} + \frac{\partial_\rho[\partial_\sigma(-g)\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})]}{(-g)16\pi G}.$$

$G^{\mu\nu}$ is the Einstein's tensor and G is the gravitational constant.

In vacuum, as in Painlevé's solution, the energy-momentum tensor is null, implying $G^{\mu\nu} = 0$, then:

$$t^{\mu\nu} = +\frac{\partial_\rho[\partial_\sigma(-g)\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})]}{(-g)16\pi G}.$$

A general analytic form is given by equation *(96,8) in [13]*

The general analytic form of this pseudo-tensor, which is given in the Landau Lifchitz [13], by the equations (96,8)[20], (96,9) shows that it is a sum of terms, all having a 1st-order derivative, of the metric, in factor[21]. Notice that there is no second-order derivative of the metric in the general analytic form. This is a structural property of this pseudo-tensor, resulting from its construction. Notice that the simplified form, in vacuum, seems to include some second-order derivative![22]

This important property implies that it will vanish in the inertial local frame, because in these coordinates, all the 1st-order derivatives of the metric vanish, therefore all of its terms will vanish. This is a very strong condition which is valid for any form of metric! But we know that these coordinates do not derive from analytic global coordinates over the manifold.

The vanishing of pseudo-tensor in Painlevé's Cartesian coordinates responds to a weaker constraint. The 1st-order derivatives of the metric, in these coordinates, do not all vanish. It is the algebraic

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[17] The covariant conservation equation $T^{\mu}{}_{\nu;\mu} = 0 \rightarrow T^{\mu}{}_{\nu;\mu} = g^{-1/2}\partial_\mu([-g^{1/2}][T^{\mu}{}_{\nu}]) - ½\partial_\nu(g_{\mu\lambda})T^{\mu\lambda} = 0,$ shows that, in the vacuum, $\partial_\mu T^{\mu}{}_{\nu}=0,$ as $T^{\mu}{}_{\nu} \equiv 0$ (except on the singularity at $r = 0$ where $T^{\mu}{}_{\nu} \equiv \infty$). This is a special case which corresponds to the case of the solution described by Painlevé's form of metric. The vanishing of this divergence should not imply the vanishing of Landau-Lifchitz's pseudo-tensor. The semi-colon symbol ";" denotes the covariant derivative and comma "," the ordinary derivative.

[18] The authors point out that the definition of a pseudo-tensor satisfying this equation, is not unique (Einstein's pseudo-tensor is also a solution) but the choice made, which contains only 1st-order derivatives of the metric tensor, is in addition symmetrical, this allowing the conservation of angular momentum.

[19] They defined their pseudo-tensor, by noticing that the covariant conservation equation in flat spacetime is satisfied in the inertial local coordinates, because, all the 1st-order derivatives of the metric vanish. In other coordinates, it is no longer true, the pseudo-fensor is introduced for restoring this property.

[20] There is a misprint in equation (96,8) , 2nd line, read $\Gamma^{k}{}_{ip}\Gamma^{p}{}_{mn}$ instead of $\Gamma^{k}{}_{lk}\Gamma^{p}{}_{mn}$ and $\Gamma^{k}{}_{mn}\Gamma^{p}{}_{lp}$ instead of $\Gamma^{p}{}_{mn}\Gamma^{p}{}_{lp}$

[21] Christoffel's symbol is: $\Gamma^{i}{}_{jk}= ½\,(g^{il})[\partial_j g_{lk}+ \partial_k g_{lj} - \partial_l g_{jk}).$

[22] As in vacuum, the Ricci tensor is null, involving additional relations, the second order derivatives will vanish, that is required for consistency with eq. (96,8)!

equation defining the pseudo-tensor which is null. Unlike the local inertial coordinates, Painlevé's coordinates are analytic all over the manifold, which is a fundamental difference.

As noted, this non-covariant approach is based on the inverse metric. As the double divergence on indices $\rho,\sigma,$ of the tensor $\lambda^{\mu\nu\rho\sigma} = K(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})$, does not formally vanish because it is not anti-symmetric on these indices, such a property can only result from symmetries, such as (anti)-auto-duality, involving very special form of the inverse metric ($g^{\mu\nu}$). The form that we have described, $\partial_s{}^2 = \delta^{ij}\,\partial_i\,\partial_j - V^\mu V^\nu\,\partial_\mu\,\partial_\nu$, where V^μ is a 4-vector which, in Cartesian coordinates, which is:

$$V^\mu = (2M)^{1/2}(x^2 + y + z^2)^{-3/4}\{\,2M^{-1/2}(x^2 + y + z^2)^{3/4}, -x, -y, -z\,\},$$

satisfies, not trivially, this condition[23].

The structure of this inverse metric tensor allowing this property is clearly exhibited when we decompose the pseudo-tensor ($t^{\mu\nu}$) into pieces: the *00* component is a scalar, the $0i = i0$ components form a 3-vector and the *ij* components form a two-index symmetric spatial 3-matrix. The vanishing of each piece may then be analyzed independently when we perform $\partial_\rho\partial_\sigma\lambda^{\mu\nu\rho\sigma}$ for getting the pseudo-tensor[24]. As noted, before, this should not be confused with the vanishing of pseudo-tensor, in the inertial local coordinates, where all the 1st-order derivatives of the metric are vanishing.

Physical meaning of the vanishing of Landau-Lifchitz's pseudo-tensor

What would be the physical meaning of the vanishing of this pseudo-tensor? As it represents the gravitation in Landau-Lifchitz's formalism, this means that gravity strength should be null!

In general relativity the (active) mass, generating the gravitational field is evaluated in terms of conserved current through spatial surfaces surrounding the central singularity (it is computed at infinity where the asymptotic flat geometry of the surface is an ordinary two-sphere). The calculation, using Landau-Lifchitz's formalism[25], shows that, in this form of Painlevé's metric, the mass of the "black hole" is zero.
This indicates that, in these coordinates, no current representing the effect of gravity, flows through these surfaces: This shows that these surfaces are "comoving" with the flow, confirming, that Painlevé's form describes a spacetime which collapses and that comoving observers, such as Painlevé's observers, do not undergo gravity.

The time coordinate is the free fall radial geodesic

It is because the time coordinate is the free fall radial incoming geodesic that we get this phenomenology.

[23] The form of Kerr-Schild, without rotation, where the 4-vector of components V_μ is null, complies also to these requirements.

[24] All these pieces derive from shift vector (β) which is radial and whose value depends only of the radial coordinate. Recall that $g = -1$ and that the metric does not depend on time (only the spatial derivatives are to be perform). The vanishing of the scalar t^{00}, coming from $\partial_\rho\partial_\sigma\lambda^{00\rho\sigma}$, is trivial because the inverse metric component $g^{00} = -1$. The matrix t^{ij}, coming from $\partial_\rho\partial_\sigma\lambda^{ij\rho\sigma}$, vanishes because of the spherical symmetry of the spatial section g^{ij} of the inverse metric. The vector part $t^{0i} = t^{i0}$, coming from $\partial_\rho\partial_\sigma\lambda^{0i\rho\sigma}$, where the 1st-divergence will yield a set of curls whose second-divergence will vanish! A similar method applies for the Kerr-Schild form, except for t^{00} whose vanishing results from the double divergence of a harmonic function.

[25] See, [9], p. 292-295, for a demonstration, confirmed by using the ADM formalism p. 324-325.

Conclusion

This contribution of Painlevé, a scientist educated in the classical Newtonian theory [26], who unwillingly would have strengthened the general relativity whose bases were still shaky at the time, essentially plagued by conceptual problems, has not been understood by his contemporaries.

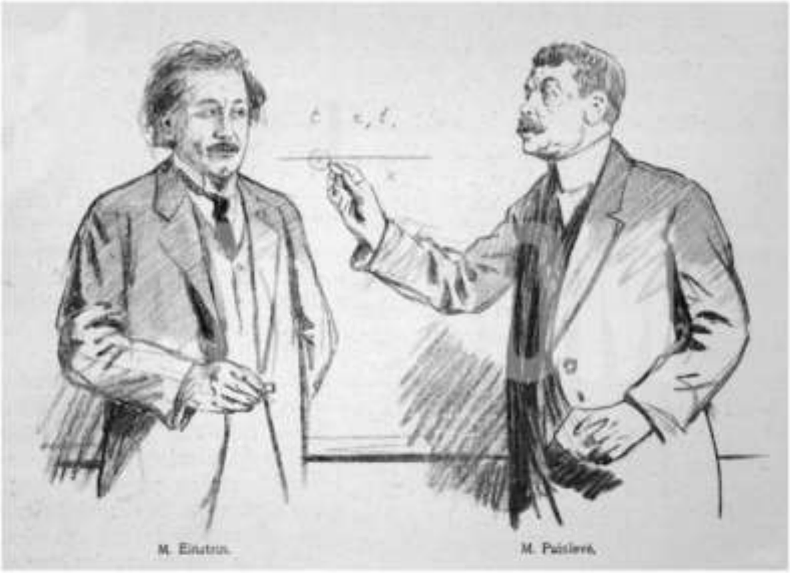
Painlevé did not understood the epistemological foundation of the general relativity, considering it as some part to be included in Newtonian mechanics. But as a mathematician this did not prevent him to understood the formalism of the theory, and to set up an innovative form of the metric who has baffled even the most brilliant minds, including Einstein, and whose merits are, at last, recognized today.

One may consider this as a happy coincidence, but one may also consider that his poor understanding of the general relativity prevented him to stick at already approved concepts and allowed his mind, free of these constraints, to be open at new ideas.

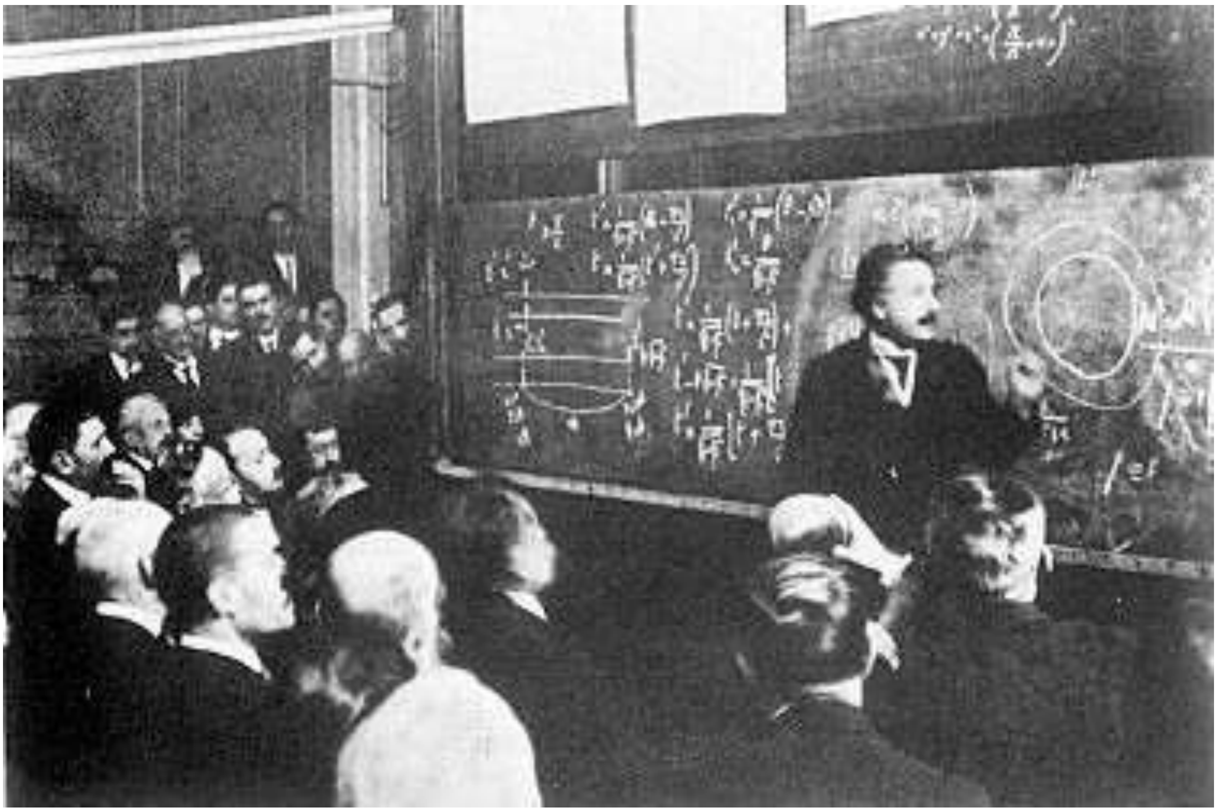
Keeping this in mind, we explored how this form enlightens the phenomenology of the solution and calls some issues, which are far from having been fully clarified even today. This demonstrates the universality of such analysis that raises issues far beyond from the original scope of the survey.

[26] And who is totally engaged in politics, at highest level (up to head of government), for 10 years, at that time!


L'ILLUSTRATION



Annex : From Top left → bottom right : 1st page of the magazine « L'illustration », drawing of the debate Einstein-Painlevé in « L'illustration (April 1922)», the crowd waiting for attending to Einstein's conference, dinner at « Polytechnique » in honor of



Einstein,. Below : Einstein's conference at « Collège de France » (April 5th 1922) about the horizon problem.

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