Spacelike geodesics in general relativity

Introduction

It is likely because one wonders how to give them a physical interpretation that this class of geodesics is scarcely studied in GR. They are useful in studying wormholes (connecting regions I and IV in Kruskal's metric) in static black holes (BH), since the worldlines joining them are spacelike. This helps us to understand the structure, not obvious, of this space-time.

https://preposterousuniverse.com/wp-...otes-seven.pdf, (p. 188-190)

Spacelike circular geodesics

To calculate spacelike geodesics in a BH, in Schwarzschild's metric, for instance, let us follow, ,https://preposterousuniverse.com/wp-... otes-seven.pdf (especially pp. 173-179)

which uses the general method, with the constants of motion (energy = E, angular momentum = L, which are conserved on a geodesic, because the metric does not depend on t or Φ).

The form obtained in (7.48) P.174 is convenient for computing circular geodesics.

We know the solutions which are the roots of an equation of the second degree. A pair of geodesics, for each value of angular momentum, one stable and one unstable. There is no solution for r < 3 GM) for circular timelike geodesics and only one unstable null geodesic, for r = 3GM.

For spacelike geodesics it must be assumed that the metric invariant is spacelike.
The constants of motion are similar but with a spacelike affine parameter λ instead of τ .

If we consider that L is real, we obtain circular geodesics of type space between r = 0 and r = 3 GM and also for r < 0, whether we suppose that it makes sense to associate it to the "White hole", whose physical existence is quite speculative, symmetrical of the black hole in the analytic maximal extension of the solution as described by the Kruskal metric, for instance.

Let us recall some key points from the above cited document.

Constants of motion

They are associated to conserved quantities on a geodesic.

1- Energy conservation: $$E=\left(1-\frac{2GM}{r}\right)\frac{dt}{d\lambda} \quad (1)$$

2- Conservation of angular momentum: $$L=r^2\frac{d\phi}{d\lambda} \quad (2)$$

3- metric invariant: $$\epsilon = -g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} \quad (3)$$

The value of ϵ is:

1, for timelike spacetime curves.
0, for null curves. (light)
-1 for spacelike curves.

Hamiltonian form of the geodesic equation

$$-E^2+\left(\frac{dr}{d\lambda}\right)^2+\left(1-\frac{2GM}{r}\right)\left(\frac{L^2}{r^2}+\epsilon\right)=0 \quad (4)$$

We may write it :

$$\frac{1}{2}E^2=\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2+V(r) \quad (5)$$

A total energy which is the sum of a potential and of a kinetic energy , with :

$$V(r)=\frac{1}{2}\epsilon-\epsilon\frac{GM}{r}+\frac{L^2}{2r^2}-\frac{GML^2}{r^3} \quad (6)$$

Solutions for circular geodesics

We will search the extrema of the potential by using the derivative of V(r) and call r_c the solutions.

$$\epsilon GM(r_c)^2-L^2(r_c)+3GML^2=0 \quad (7)$$

This second-degree equation has two roots. The case of timelike and null geodesics is fully developed in the cited document. This is for:

ϵ=1, ϵ=0 (8)

For $\epsilon=-1$, we get:

$$-GMr_c^2-L^2r_c+3GML^2=0 \quad (9)$$

The solutions are :

$$r_c=\frac{-L^2+/-\sqrt{L^4+12G^2M^2L^2}}{2GM} \quad (10)$$

Comments on these solutions

Positive root

a- **We will get 2GM < r_c < 3 GM pour L > 1**

This part of spacetime is between the horizon and the photon circular unstable geodesic, where no timelike circular geodesic exists. This is consistent with the timelike analysis.
Let us stress that circular non geodesic timelike worldlines may exist in this part of spacetime.
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b- **We will get 0< r_c <2 GM, pour L < 1**

This part of spacetime is under the horizon where no circular timelike worldline exist. All timelike worldline would reach the central singularity.

Negative root

The negative root provides a solution where: 0 > rc > -∞.

Whether we interpret it as a solution in the anti-universe (the white hole, region IV in Kruskal metric) where the coordinate *r* would be negative, it would be consistent with the phenomenology of the region IV where « gravity » would be repulsive. But this is subject to the physical existence of this anti-universe?

Length of a circular spacelike geodesic

From equation (2) $L = r^2 \frac{d\phi}{d\lambda}$

We deduce $d\lambda = \frac{r^2 d\phi}{L}$ (11)

The length Λ of the circular spacelike geodesic is given by integrating *Φ* from*0* to *2π* :

$$\Lambda = \frac{2\pi r^2}{L}$$

By using *λ* as spacelike affine parameter and equation (9) for getting the value of L^2 as a function of *r* which is:

$$L^2 = \frac{GM.r^2}{3GM - r} \qquad (12)$$

We get:

$$\Lambda = \frac{2\pi r^2 \sqrt{3GM - r}}{\sqrt{GMr^2}} \qquad (13)$$

Finally, by simplifying**:** $$\boldsymbol{\Lambda = \frac{2\pi r \sqrt{3GM - r}}{\sqrt{GM}}} \qquad \textbf{(14)}$$

Let us stress that this equation is defined for r< 3GM and that negative values of *r* are (mathematically) valid. One can also compute the parameter T associated to the coordinate *t* for this geodesic, from the metric spacelike invariant.

$$-1 = \left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{L^2}{r^2} \longrightarrow dt^2 = \left(\frac{L^2}{r^2} - 1\right)\left(\frac{r}{r-2GM}\right) d\lambda^2 \tag{15}$$

Substituting to L^2 its value given by (12) and simplifying gives

$$dt^2 = \left(\frac{GM.r^2}{(3GM-r)r^2} - 1\right)\left(\frac{r}{r-2GM}\right) d\lambda^2 = \left(\frac{GM-(3GM-r)}{(3GM-r)}\right)\left(\frac{r}{r-2GM}\right) d\lambda^2 =$$
$$\left(\frac{r-2GM)}{(3GM-r)}\right)\left(\frac{r}{r-2GM}\right) d\lambda^2 = \left(\frac{r}{(3GM-r)}\right) d\lambda^2 \tag{15}$$

By integrating on the circle of the spacelike geodesic, for Φ from 0 to 2π, provides a «pseudo- period» T'

$$T' = \frac{r}{3GM-r}\,\Lambda = \frac{2\pi\, r\sqrt{r}}{\sqrt{GM}} \tag{16}$$

That's the Kepler's law, as for timelike geodesics but let us notice that it is real only for $r \geq 0$, because of the square root of r in the equation

Comparison with the timelike geodesic.

All operations are the same that these of spacelike circular geodesics but with ∈ = 1.

For a timelike circular geodesic with τ as affine parameter, by the same method on can verify that, the period on the circular geodesic is :

$$\boldsymbol{\tau = \frac{2\pi r\sqrt{r-3GM}}{\sqrt{GM}}} \tag{14'}$$

The equation gives a real result for r > 3GM, in accordance with well-known results.

For the pseudo-period with coordinate t as parameter we get:

$$T = \frac{r}{r-3GM}\,\tau = \frac{2\pi\, r\sqrt{r}}{\sqrt{GM}} \tag{16'}$$

This is also the Kepler's law which is valid as from r =0, per the Newton mechanics unlike the equation in general relativity where timelike circular geodesic starts as from r > 3 GM. This shows that they are physical different theories.